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A STUDY OF LONGITUDINAL OSCILLATIONS OF PROPELLANT TANKS AND WAVE PROPAGATIONS IN FEED LINES

Part III - Wave Propagation in an Elastic Pipe Filled With Incompressible Viscous Streaming Fluid

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FOREWORD

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Part I - One-Dimensional Wave Propagation in a Feed Line

Part II - Wave Propagation in an Elastic Pipe Filled With Incompressible Viscous Fluid

Part III - Wave Propagation in an Elastic Pipe Filled With Incompressible Viscous Streaming Fluid

Part IV - Longitudinal Oscillation of a Propellant-Filled Flexible Hemispherical Tank

Part V - Longitudinal Oscillation of a Propellant-Filled Flexible Oblate Spheroidal Tank

The project was carried out by the Launch Vehicle Dynamics Group, Structures and Dynamics Department of Research and Engineering Division, S&ID. Dr. F.C. Hung was the Program Manager for North American Aviation, Inc. The study was conducted by Dr. Clement L. Tai (Principal Investigator), Dr. Michael M.H. Loh, Mr. Henry Wing, Dr. Sui-An Fung, and Dr. Shoichi Uchiyama. Dr. James Sheng, who started the investigation of Part IV, left in the middle of the program to teach at the University of Wisconsin. The computer program was developed by Mr. R.A. Pollock, Mr. F.W. Egeling, and Mr. S. Miyashiro.

ABSTRACT

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The study of the longitudinal wave propagation in an elastic pipe filled with imcompressible viscous fluid at rest has been extended to include the effect of steady stream.

The Navier-Stokes equations are first introduced for an axisymmetric and incompressible flow. Linearization and simplification are obtained by the assumptions of small disturbance and small fluid viscosity. By linearizing the boundary conditions, the equations of motion of the pipe are greatly simplified and become linear. Perturbation technique is applied; the perturbed equations are solved by two transformations and the boundary layer method. Finally, based on the equations of motion of pipe and fluid and proper boundary conditions, a characteristic equation is formed in terms of phase velocity ratio and four independent parameters — viscosity parameter, steady-state velocity parameter, and the axial and radial inertia parameters. This fourth-order complex-valued characteristic equation is then separated, corresponding to real and imaginary parts, into two equations that are solved by digital computing. The method of numerical analysis and computing procedures are described in detail and some results are presented.

CONTENTS

ILLUSTRATIONS

NOMENCLATURE

A_1	Constant
B_1, C, C'	Constants
c	Phase velocity
c_o	Phase velocity in fluid at rest
e	Root-mean-square error of e_1 and e_2
e_1	Error from first equation
e_2	Error from second equation
E	Young's modulus of materials
h	Thickness of pipe wall
i	$\sqrt{-1}$
I	Wall inertia factor, $I = \dfrac{(1 - \sigma^2)\ \rho_o \omega^2}{Ek^2}$
k	Complex propagation constant, $k = k_1 + ik_2$
k_1	Phase factor
k_2	Attenuation factor
K_1	Phase velocity parameter, $K_1 = \dfrac{k_1}{\omega}\sqrt{\dfrac{Eh}{2\rho R_o}}$
K_2	Attenuation parameter, $K_2 = \dfrac{k_2}{\omega}\sqrt{\dfrac{Eh}{2\rho R_o}}$
L	Length of pipe

n	Constant
$\underline{\underline{o}}$	Indicates order of magnitude
p	Pressure
P	Constant (refer to Equation (3-24)
Q	Volumetric flow rate
r	Coordinate in radial direction
t	Time
T	Tensile force (refer to Equations (2-6) and (2-7))
u	Displacement of pipe wall in axial direction
U	Constant
U^s	Average steady-state velocity
v	Fluid velocity component
V	Function of r (refer to Equation 3-22)
w	Displacement of pipe wall in radial direction
W	Constant (refer to Equation 3-25)
x	Coordinate in axial direction
X	Function of r (refer to Equation 3-21)
y	Radius ratio, $y = \frac{r}{R_o}$
z	Viscosity parameter, $z = R_o \left(\frac{\rho \omega}{\mu}\right)^{1/2}$
$\alpha_{11}, \alpha_{12}, \alpha_{21}, \alpha_{22}$	Coefficients (refer to Equation 3-45)
β	Axial inertia parameter, $\beta = \frac{(1 - \sigma^2) \rho_o h}{2 \rho R_o}$

γ	Steady-state velocity parameter, $\gamma = \dfrac{U^s}{\sqrt{\dfrac{Eh}{2\rho R_o}}}$
δ	Small distance
ϵ	Dimensionless number, $\epsilon = \dfrac{\mu}{\rho\omega R_o^2} = z^{-2}$
η	Radial inertia parameter, $\eta = \dfrac{\rho_o \omega^2 R_o^2}{E}$
$2\pi\lambda$	Wave length
μ	Dynamic viscosity
ρ	Mass density
σ	Poisson's ratio
τ	Stress component
ϕ	Function of y (refer to Equation (3-32))
Φ	Function of Ω (refer to Equation (3-36))
ψ	Stream function
Ψ	Function of r (refer to Equation 3-29)
χ	Phase velocity ratio, $\chi = \dfrac{k}{\omega}\sqrt{\dfrac{Eh}{2\rho R_o}}$
ω	Circular frequency of disturbance
Ω	Dimensionless independent variable, $\Omega = \dfrac{1-y}{\epsilon^n} = z(1-y)$ (refer to Equation (3-36))

Superscripts

s	Steady stream
'	Perturbed stream

Subscripts

0	Unstressed
1	Surrounding materials
θ	Circumferential direction
x	Axial direction
r	Radial direction

1. INTRODUCTION

The problem of wave propagation in an elastic pipe filled with incompressible and viscous fluid was discussed in Part II of this report (Reference 1). The work discussed in this part extends the investigation to a more general situation in which the pressure waves travel through a system filled with streaming fluid.

The propagation of pressure waves through liquid-filled elastic pipes has been investigated by many authors; attempts to take account of the steady stream have also been made by others (References 2, 3, 4, 5, and 6). In the previous works, either the motion was restricted to parallel flow or the average stream velocity was used instead of the actual nonuniform velocity, as has been discussed in Part II of this report. The present investigation set out with the classical Navier-Stokes equations of two-dimensional flow with axisymmetric motion. The elastic equations of the pipe are derived on the basis of a thin shell. These equations are simplified by omitting terms of small order of magnitude. By means of the perturbation technique and boundary layer analysis based on the smallness of the parameter $(1/R_o)(\mu/\omega\rho)^{1/2}$, a characteristic equation is obtained. The general approach to the solution follows closely the procedures of Morgan and Ferrante (Reference 5), who merit full credit of their excellent work. The terms contributed by the radial pipe wall inertial forces, which were omitted by Morgan and his associate, are retained in this analysis. In the case of wave propagation of light-weight fluid such as liquid hydrogen through a metallic tube, it is important to examine the effect of both these inertial forces.

The characteristic equation is being solved numerically with the aid of a digital computer. It is hoped that the results of the numerical analysis will reveal the relations among the various parameters and the influence of the steady stream upon which waves are superposed. Due to the complex nature of the characteristic equations, at the time this report was prepared no conclusive results comparable to those of wave propagation through fluid at rest have been obtained. However, the analytical procedures are discussed fully and partial results are presented.

2. BASIC EQUATIONS

The Navier-Stokes equations expressed in cylindrical coordinates with the assumptions of incompressible fluid and axisymmetrical motion are

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + v_x \frac{\partial v_r}{\partial x}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{v_r}{r^2} + \frac{\partial^2 v_r}{\partial x^2}\right) \tag{2-1}$$

$$\rho\left(\frac{\partial v_x}{\partial t} + v_r \frac{\partial v_x}{\partial r} + v_x \frac{\partial v_x}{\partial x}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial r^2} + \frac{1}{r}\frac{\partial v_x}{\partial r} + \frac{\partial^2 v_x}{\partial x^2}\right) \tag{2-2}$$

Here ρ and μ are the mass density and dynamic viscosity of the fluid, respectively; p is the pressure; v_x and v_r are the velocity components of fluid in the axial and radial directions, respectively.

The continuity equation is

$$\frac{\partial v_x}{\partial x} + \frac{v_r}{r} + \frac{\partial v_r}{\partial r} = 0 \tag{2-3}$$

The normal and shear forces acting on a unit area perpendicular to the radius are given by

$$\tau_{rr} = p - 2\mu \frac{\partial v_r}{\partial r} \tag{2-4}$$

$$\tau_{rx} = \mu\left(\frac{\partial v_x}{\partial r} + \frac{\partial v_r}{\partial x}\right) \tag{2-5}$$

For a thin elastic pipe, the hoop tension, T_θ, and the tensile force, T_x, respectively, are

$$T_\theta = \frac{Eh}{1-\sigma^2}\left(\frac{w}{R_o} + \sigma \frac{\partial u}{\partial x}\right) \tag{2-6}$$

$$T_x = \frac{Eh}{1 - \sigma^2}\left(\frac{\partial u}{\partial x} + \sigma \frac{w}{R_o}\right) \tag{2-7}$$

where u and w are the axial and radial components of displacement of the pipe wall. Then the equations of motion of the pipe are given in the following forms:

$$\rho_o h \frac{\partial^2 w}{\partial t^2} = \tau_{rr}(R, x, t) - \frac{Eh}{1 - \sigma^2}\left(\frac{w}{R_o^2} + \frac{\sigma}{R_o}\frac{\partial u}{\partial x}\right) \tag{2-8}$$

$$\rho_o h \frac{\partial^2 u}{\partial t^2} = - \tau_{rx}(R, x, t) + \frac{Eh}{1 - \sigma^2}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\sigma}{R_o}\frac{\partial w}{\partial x}\right) \tag{2-9}$$

where ρ_o and h are the mass density and thickness of pipe wall, respectively.

3. PERTURBATION METHOD

Since we are investigating the problem of pressure wave propagation superposed on a steady stream, we assume all the dependent variables to have the form

$$u = u^s + u' \tag{3-1}$$

$$w = w^s + w' \tag{3-2}$$

$$v_x = v_x^s + v_x' \tag{3-3}$$

$$v_r = v_r^s + v_r' \tag{3-4}$$

$$p = p^s + p' \tag{3-5}$$

where the superscripts s and prime refer to steady and perturbed quantities.

Applying the perturbation solutions, Equations (2-1) and (2-2) yield two sets of equations for the steady and the perturbed motion. For the steady state,

$$\rho\left(v_r^s \frac{\partial v_r^s}{\partial r} + v_x^s \frac{\partial v_r^s}{\partial x}\right) = -\frac{\partial p^s}{\partial r} + \mu\left(\frac{\partial^2 v_r^s}{\partial x^2} + \frac{\partial^2 v_r^s}{\partial r^2} + \frac{1}{r}\frac{\partial v_r^s}{\partial r} - \frac{v_r^s}{r^2}\right) \tag{3-6}$$

$$\rho\left(v_r^s \frac{\partial v_x^s}{\partial r} + v_x^s \frac{\partial v_x^s}{\partial x}\right) = -\frac{\partial p^s}{\partial x} + \mu\left(\frac{\partial^2 v_x^s}{\partial r^2} + \frac{1}{r}\frac{\partial v_x^s}{\partial r} + \frac{\partial^2 v_x^s}{\partial x^2}\right) \tag{3-7}$$

For the perturbed state, with the nonlinear terms of primed quantities omitted,

$$\rho\left(\frac{\partial v_r'}{\partial t} + v_r' \frac{\partial v_r^s}{\partial r} + v_r^s \frac{\partial v_r'}{\partial r} + v_r' \frac{\partial v_r'}{\partial r} + v_x' \frac{\partial v_r^s}{\partial x} + v_x^s \frac{\partial v_r'}{\partial x} + v_x' \frac{\partial v_r'}{\partial x}\right)$$

$$= -\frac{\partial p'}{\partial r} + \mu\left(\frac{\partial^2 v_r}{\partial x^2} + \frac{\partial^2 v_r'}{\partial r^2} + \frac{1}{r}\frac{\partial v_r'}{\partial r} - \frac{v_r'}{r^2}\right) \tag{3-8}$$

$$\rho\left(\frac{\partial v'_x}{\partial t} + v^s_r \frac{\partial v'_x}{\partial r} + v'_r \frac{\partial v^s_x}{\partial r} + v'_r \frac{\partial v'_x}{\partial r} + v'_x \frac{\partial v^s_x}{\partial x} + v^s_x \frac{\partial v'_x}{\partial x} + v'_x \frac{\partial v'_x}{\partial x}\right)$$

$$= -\frac{\partial p'}{\partial x} + \mu\left(\frac{\partial^2 v'_x}{\partial r^2} + \frac{1}{r}\frac{\partial v'_x}{\partial r} + \frac{\partial^2 v'_x}{\partial x^2}\right) \tag{3-9}$$

If we linearize the boundary conditions at the interface of pipe and fluid by evaluating parameters at R_o rather than at the true radius R, the Equations (2-8) and (2-9) become linear. Then the elastic equations (2-8) and (2-9) and the continuity equation (2-3) apply to both the steady and perturbed states with appropriate superscripts.

STEADY-STATE SOLUTION

The investigation commences with the assumption of steady-state flow. Since it is stipulated that the slope of the disturbed pipe wall is very small, the radial velocity, v^s_r, and its derivatives may be expected to be small. From Equation (3-6), it may be said that p is a function of x only and the continuity equation, $\partial v^s_x/\partial x$ becomes very small, then Equation (3-7) becomes

$$\frac{d}{dr}\left(r\frac{dv^s_x}{dr}\right) = \frac{r}{\mu}\frac{dp^s}{dx} \tag{3-10}$$

Integrating Equation (3-10) with proper boundary conditions, the velocity profile is obtained:

$$v^s_x = -\frac{(R^s)^2}{4\mu}\frac{dp^s}{dx}\left[1 - \left(\frac{r}{R^s}\right)^2\right] \tag{3-11}$$

If the average velocity and volumetric flow rate are denoted by U^s and Q respectively, the pressure gradient may be expressed as

$$\frac{dp^s}{dx} = -\frac{8\mu Q}{\pi(R^s)^4} = -\frac{8\mu}{(R^s)^2}U^s \tag{3-12}$$

Neglecting v^s_r, Equations (2-8) and (2-9) of steady flow become

$$p^s = \frac{Eh}{1-\sigma^2}\left(\frac{w^s}{R_o^2} + \frac{\sigma}{R_o}\frac{\partial u^s}{\partial x}\right) \tag{3-13}$$

$$\mu \frac{\partial v_x^s}{\partial r}\Bigg]_{r = R^s} = \frac{Eh}{1 - \sigma^2}\left(\frac{\partial^2 u^s}{\partial x^2} + \frac{\sigma}{R_o}\frac{\partial w^s}{\partial x}\right) \qquad (3\text{-}14)$$

Differentiating Equation (3-13) with respect to x and combining with Equations (3-12) and (3-14) gives

$$\frac{\partial w^s}{\partial x} = \frac{-4\,\mu\, U^s}{Eh}\left(\frac{R_o}{R^s}\right)^2 \left[2 - \sigma\left(\frac{R^s}{R_o}\right)\right] = \frac{dR^s}{dx} \qquad (3\text{-}15)$$

Integrating from x = 0 to x = L, gives

$$w^s_{x = L} = w_{x = 0} - \frac{4(2 - \sigma)\,\mu U^s}{Eh}\, L \qquad (3\text{-}16)$$

If $p_{x=0}$ is denoted as the average pressure at this entrance section of the pipe where x = 0 and $u^s = 0$, by stress-strain relation, $w_{x = 0}$ may be expressed as

$$w_{x = 0} = \frac{(p_{x = 0})\, R_o^2}{Eh} \qquad (3\text{-}17)$$

Since v_x^s and v_r^s may be written in terms of R^s and R^s in terms of x (refer to Equations (3-11) and (3-15)), then terms like $\partial v_x^s/\partial x$ and $\partial v_r^s/\partial x$ may be evaluated. Thus, by using the continuity Equation (2-3), and hydrodynamic Equations (2-1), we have

$$v_r^s = \frac{-16\mu Q^2 R_o^2}{\pi^2 Eh (R^s)^7}\left[r\left(1 - \frac{\sigma R^s}{2R_o}\right) - \left(\frac{1}{R^s} - \frac{\sigma}{2R_o}\right)\frac{r^3}{R^s}\right] \qquad (3\text{-}18)$$

$$p^s = \frac{32\mu^2 (U^s)^2 (R_o)^2}{Eh (R^s)^5}\left(2 - \sigma\frac{R^s}{R_o}\right) r^2 + p_{x=0} \qquad (3\text{-}19)$$

The axial displacement, u^s, is found through Equation 3-13,

$$u^s = \int_0^x \left(\frac{1 - \sigma^2}{Eh}\, p^s - \frac{w^s}{R_o^2}\right)\frac{R_o}{\sigma}\, dx \qquad (3\text{-}20)$$

THE PERTURBATION EQUATIONS

The pertinent equations of the liquid motion are Equations (3-8) and (3-9). The investigation will be limited to a disturbance that is harmonic in time and of small amplitude. By the first limitation, v'_x and v'_r may be expressed as

$$v'_x = X(r) \exp\left[i\,(kx - \omega t) \right] \tag{3-21}$$

$$v'_r = V(r) \exp\left[i\,(kx - \omega t) \right] \tag{3-22}$$

where X and V are functions of r, ω is the circular frequency of the forced disturbance, k is a complex propagation constant and may be expressed as $k_1 + ik_2$, of which k_1 is the phase factor representing the phase shift and k_2 is an attenuation factor representing a measure of the decay of the disturbance as the wave travels along the pipe. Because of the second limitation, the nonlinear terms of the primed quantities in Equations (3-8) and (3-9) become negligibly small and are dropped from the equations.

Further simplification of the equation may be obtained by dropping the terms $v^s_r(\partial v'_x/\partial r)$, $v'_x(\partial v^s/\partial r)$ and $\partial^2 v'_x/\partial x^2$, because the relative magnitudes of these terms are much smaller than the corresponding terms $v'_r(\partial v^s_x/\partial r)$, $v^s_x(\partial v'_x/\partial x)$, and $\partial^2 v'/\partial r^2$. Thus, Equation (3-9) becomes

$$\rho\left(\frac{\partial v'_x}{\partial t} + v'_r\,\frac{\partial v^s_x}{\partial r} + v^s_x\,\frac{v'_x}{\partial x}\right) = -\frac{\partial p'}{\partial x} + \mu\left(\frac{\partial^2 v'_x}{\partial r^2} + \frac{1}{r}\,\frac{\partial v'_x}{\partial r}\right) \tag{3-23}$$

Since v'_r and its derivatives are expected to be small, as mentioned before, from Equation (3-8), $\partial p'/\partial r = 0$; therefore, we may write

$$p' = P \exp\left[i\,(kx - \omega t) \right] \tag{3-24}$$

consequently, for a thin-wall pipe, we may also write

$$w' = W \exp\left[i\,(kx - \omega t) \right] \tag{3-25}$$

$$u' = U \exp\left[i\,(kx - \omega t) \right] \tag{3-26}$$

where P, W and U are all constants. Then the equations of motion of the pipe wall become

$$-\frac{Eh}{1-\sigma^2}\left(\frac{W}{R_o^2} + i\frac{\sigma}{R_o}kU\right) + \left(P - 2\mu\frac{\partial V}{\partial r}\right) = -\rho_o h\omega^2 W$$

$$\frac{Eh}{1-\sigma^2}\left(-k^2U + i\frac{\sigma}{R_o}kW\right) - \mu\left(\frac{\partial X}{\partial r}\right) = -\rho_o h\omega^2 U$$

Solving these two equations, gives

$$U = \frac{i\sigma}{kR_o}\frac{W}{1-I} - \frac{\mu(1-\sigma^2)}{Ehk^2(1-I)}\left(\frac{dX}{dr}\right)\Bigg]_{r=R_o} \qquad (3\text{-}27)$$

$$P = \frac{EhW}{R_o^2(1-I)} - \frac{EhWI}{(1-\sigma^2)R_o^2(1-I)} - \frac{i\mu\sigma}{R_o k(1-I)}\left(\frac{dX}{dr}\right)\Bigg]_{r=R_o} - \rho_o h\omega^2 W \qquad (3\text{-}28)$$

where

$$I = \frac{\rho_o \omega^2 (1-\sigma^2)}{Ek^2}$$

and is called the wall inertia factor. The last term, $\rho_o h\omega^2 W$, representing the influence of the radial wall inertia.

SOLUTION TO THE PERTURBED EQUATIONS

Introducing the stream function $\psi'(x, r, t)$ and letting $\psi' = \Psi(r) \cdot \exp\left[i(kx - \omega t)\right]$, Equation (3-23) becomes

$$\frac{d^3\Psi}{dr^3} - \frac{1}{r}\frac{d^2\Psi}{dr^2} + \left\{\frac{1}{r^2} + \frac{i\rho}{\mu}\left[\omega - 2kU^s\left(1 - \frac{r^2}{(R^s)^2}\right)\right]\right\}\frac{d\Psi}{dr}$$

$$-\frac{4i\rho\, krU^s}{\mu(R^s)^2}\Psi + \frac{ikr}{\mu}P = 0 \qquad (3\text{-}29)$$

The relationships between v'_x, v'_r and ψ' are $v'_x = (-1/r)(\alpha\psi'/\alpha r)$ and $v'_r = (1/r)(\alpha\psi'/\alpha x)$. In addition,

$$\left.\begin{aligned} X(r) &= (-1/r)(\alpha\psi/\alpha r) \\ V(r) &= i\frac{k}{r}\psi \end{aligned}\right\} \quad (3\text{-}30)$$

The nonslip boundary condition at the pipe wall may be written as

$$v'_x(R_o, x, t) = \frac{\partial u'}{\partial t}$$

$$v'_r(R_o, x, t) = \frac{\partial w'}{\partial t}$$

Thus,

$$\left.\begin{aligned} X(R_o) &= -i\omega U \\ V(R_o) &= -i\omega W \end{aligned}\right\} \quad (3\text{-}31)$$

Using Equations (3-27), (3-28), and (3-29) together with the boundary Equation (3-31), the three unknowns U, W and Ψ can be solved.

Since the viscosity is assumed to be small, the technique customarily applied in boundary layer analysis may be used to solve the present problem. Letting $y = r/R_o$ and $\Psi = C\phi(y)$ where C is a dimensioned constant, then Equation (3-29) becomes

$$\epsilon\frac{d^3\phi}{dy^3} - \frac{\epsilon}{y}\frac{d^2\phi}{dy^2} + \left\{\frac{\epsilon}{y^2} + i\left[1 - \frac{2\,kU^s}{\omega}(1-y^2)\right]\right\}\frac{d\phi}{dy} - \frac{4ikU^s y}{\omega}\phi$$

$$+ \frac{ikR_o^2 P}{\omega\rho C}\, y = 0 \qquad (3\text{-}32)$$

where $\epsilon = \mu/\rho\omega R_o^2$, a dimensionless quantity that is assumed to be small. The constant C can be so selected as to make the last term dimensionless. Such a C is $C = (\omega/k)(WR_o)$.

Since ϵ is small, the influence of the viscosity will be limited to a narrow neighborhood near the wall. Therefore, it is expected that $\epsilon\phi$ and $(\epsilon d^n\phi/dy^n)$ are negligible outside the boundary layer, compared with ϕ and

$(d^n \phi / dy^n)$. But this is not true near the wall. Let $\phi = \phi_o + \phi_c$, where ϕ_o is the solution of ϕ away from the wall, and ϕ_c is the correcting term of ϕ within the boundary layer. By so doing, two equations result: first, for the flow region outside the boundary layer,

$$\left[1 - 2\frac{k}{\omega}U^s (1 - y^2)\right] \frac{d\phi_o}{dy} - 4\left(\frac{k}{\omega}\right)U^s y\phi_o + \left(\frac{k}{\omega}\right)^2 \frac{R_o Py}{\rho W} = 0 \qquad (3\text{-}33)$$

Its solution is

$$\phi_o = C' \left[1 - 2\frac{k}{\omega}U^s (1 - y^2)\right] + \frac{k}{\omega} \frac{R_o P}{4U^s \rho W} \qquad (3\text{-}34)$$

where C' is an arbitrary constant.

Second, for the flow region inside the boundary layer.

$$\epsilon \frac{d^3\phi_c}{dy^3} - \frac{\epsilon}{y}\frac{d^2\phi_c}{dy^2} + \left\{\frac{\epsilon}{y^2} + i\left[1 - \frac{2kU^s}{\omega}(1 - y^2)\right]\right\}\frac{d\phi_c}{dy}$$

$$- 4i\frac{kU^s}{\omega} y\phi_c = 0 \qquad (3\text{-}35)$$

Before the solution is attempted, a new independent variable Ω and a new dependent variable Φ will be introduced into the above equation. Let $\Omega = (1/\epsilon^n)(1 - y)$ where n is a positive number and $\Phi(\Omega) = \phi_c(y)$. Then, Equation (3-35) becomes

$$\left(\epsilon^{1-3n}\right)\frac{d^3\Phi}{d\Omega^3} + \left(\frac{\epsilon^{1-2n}}{1 - \epsilon^n\Omega}\right)\frac{d^2\Phi}{d\Omega^2} + \left\{\frac{\epsilon}{(1 - \epsilon^n\Omega)^2} + i\left[1 - \frac{2kU^s}{\omega}\left(2\epsilon^n\Omega - \epsilon^{2n}\Omega^2\right)\right]\right\}$$

$$\left(\frac{1}{\epsilon^n}\right)\frac{d\Phi}{d\Omega} + 4i\frac{k}{\omega}U^s(1 - \epsilon^n\Omega)\Phi = 0 \qquad (3\text{-}36)$$

From an examination of the coefficients of $d^n\Phi/d\Omega^n$, it is obvious that the first and third terms are larger than the second and fourth. The smaller ones are dropped and the first and third terms assumed to be of the same order of magnitude so as to balance each other. Thus we have $1 - 3n = -n$, or $n = 1/2$. Equation (3-36) becomes

$$\frac{d^3\Phi}{d\Omega^3} + i\frac{d\Phi}{d\Omega} = 0 \qquad (3\text{-}37)$$

The solution of this equation, then, is

$$\Phi = A_1 \cdot \exp(i\sqrt{i}\,\Omega) + A_2 \cdot \exp(-i\sqrt{i}\,\Omega) + A_3$$

The condition placed on ϕ_c and hence on Φ is that Φ approaches zero as r approaches zero. This is possible only if $A_2 = A_3 = 0$.

From the relationship between Φ and Ψ, an expression of Ψ is obtained in terms of the two unknown constants, A_1 and C'. However, C' may be determined by using Equation (3-30); that is, $V(r)$ or $(i\,k/r)\,\Psi$ must vanish at $r = 0$. Thus we have

$$C' = -\frac{k}{\omega}\,\frac{R_o P}{4U^s \rho W}\left(1 - 2\frac{k}{\omega}U^s\right)^{-1}$$

If a new constant, B_1, is defined containing the remaining unknown, A_1, as

$$B_1 = A_1 \frac{\omega}{k} W R_o$$

then

$$\Psi = \frac{-k}{\omega}\,\frac{Pr^2}{2\rho\left(1 - 2\frac{k}{\omega}U^s\right)} + B_1 \cdot \exp\left[i\sqrt{i}\,\epsilon^{-\frac{1}{2}}\left(1 - \frac{r}{R_o}\right)\right] \qquad (3\text{-}38)$$

Applying Equation (3-38) to (3-30) gives

$$X(r) = \frac{k}{\omega} \frac{P}{\rho\left(1 - 2\frac{k}{\omega}U^s\right)} + \frac{B_1}{rR_o} i\sqrt{i}\,\epsilon^{-\frac{1}{2}} \exp\left[i\sqrt{i}\,\epsilon^{-\frac{1}{2}}\left(1 - \frac{r}{R_o}\right)\right] \quad (3\text{-}39a)$$

$$V(r) = \frac{-ik^2}{\omega} \frac{Pr}{2\rho\left(1 - 2\frac{k}{\omega}U^s\right)} + \frac{ikB_1}{r} \cdot \exp\left[i\sqrt{i}\,\epsilon^{-\frac{1}{2}}\left(1 - \frac{r}{R_o}\right)\right] \quad (3\text{-}39b)$$

Evaluating Equations (3-39) at $r = R_o$ and comparing the results with (3-31) leads to Equations (3-40) and (3-41).

$$\frac{k}{\omega} \frac{P}{\rho\left(1 - 2\frac{k}{\omega}U^s\right)} + \frac{i\sqrt{i}\,\epsilon^{-\frac{1}{2}}}{R_o^2} B_1 + i\omega U = 0 \quad (3\text{-}40)$$

$$\left(\frac{k}{\omega}\right)^2 \frac{R_o P}{2\rho\left(1 - 2\frac{k}{\omega}U^s\right)} - \frac{k}{\omega}\frac{B_1}{R_o} - W = 0 \quad (3\text{-}41)$$

To determine U and P from Equations (3-27) and (3-28).

$$\left.\frac{dX}{dr}\right]_{r = R_o}$$

must be evaluated first. From Equation (3-39a),

$$\left.\frac{dX}{dr}\right]_{r = R_o} = \frac{B_1}{R_o^3}\left\{\frac{i}{\epsilon} + \frac{1 - i}{\sqrt{2\epsilon}}\right\} \quad (3\text{-}42)$$

The last two terms can be dropped out; they are insignificant because the boundary layer analysis can be expected to yield a good approximation only to the first power of ϵ.

Substituting

$$\left.\frac{dX}{dr}\right]_{r = R_o} = \frac{B_1 i}{R_o^3 \epsilon}$$

into Equations (3-27) and (3-28), we have

$$U = \frac{i \sigma W}{k R_o (1 - I)} - \frac{i (1 - \sigma^2) \mu B_1}{Ehk^2 (1 - I) \epsilon R_o^3} \tag{3-43}$$

$$P = \left[\frac{Eh}{(1 - I) R_o^2} - \frac{EhI}{(1 - \sigma^2)(1 - I) R_o^2} - \rho_o h \omega^2 \right] W + \frac{\sigma \mu B_1}{k \epsilon R_o^4 (1 - I)} \tag{3-44}$$

Substituting the above values of U and P into Equations (3-40) and (3-41), two simultaneous equations result:

$$\alpha_{11} W + \alpha_{12} B_1 = 0$$

$$\alpha_{21} W + \alpha_{22} B_1 = 0$$

This system will have nontrivial solutions only if the determinant of the coefficients vanishes; that is

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{vmatrix} = 0$$

This gives

$$\alpha_{11} \alpha_{22} - \alpha_{12} \alpha_{21} = 0 \tag{3-45}$$

$$\alpha_{11} = -\left(\frac{k}{\omega}\right)^3 \left(\frac{Eh}{2\rho R_o}\right)^2 R_o \left(1 - 2\frac{k}{\omega}U^s\right)(1 - I)$$

$$+ \frac{IR_o}{1-\sigma^2}\left(\frac{k}{\omega}\right)^3\left(\frac{Eh}{2\rho R_o}\right)^2\left(1 - 2\frac{k}{\omega}U^s\right)(1 - I)$$

$$+ \left[\left(\frac{\rho_o h\omega^2}{2\rho}\right)^2\left(\frac{k}{\omega}\right)^3\left(\frac{Eh}{2\rho R_o}\right)(R_o)^2\left(1 - 2\frac{k}{\omega}U^s\right)(1 - I)^2\right]$$

$$+ \left[\frac{\sigma R_o}{2}\left(\frac{k}{\omega}\right)\left(\frac{Eh}{2\rho R_o}\right)\left(1 - 2\frac{k}{\omega}U^s\right)^2(1 - I)\right] \tag{3-45a}$$

$$\alpha_{12} = \left(\frac{-i\sqrt{i}}{2}\sqrt{\frac{\omega\rho}{\mu}}\right)\left(\frac{k}{\omega}\right)^2\left(\frac{Eh}{2\rho R_o}\right)(R_o)\left(1 - 2\frac{k}{\omega}U^s\right)^2(1 - I)^2$$

$$-\left[\frac{1}{4}(1-\sigma^2)\left(1 - 2\frac{k}{\omega}U^s\right)^2(1 - I)\right]$$

$$-\left[\frac{\sigma}{2}\left(\frac{k}{\omega}\right)^2\left(\frac{Eh}{2\rho R_o}\right)\left(1 - 2\frac{k}{\omega}U^s\right)(1 - I)\right] \tag{3-45b}$$

$$\alpha_{21} = \left(\frac{k}{\omega}\right)^2\left(\frac{Eh}{2\rho R_o}\right)\left(1 - 2\frac{k}{\omega}U^s\right)^{-1}(1 - I)^{-1} - 1$$

$$-\frac{I}{(1-\sigma^2)(1 - I)}\left(\frac{k}{\omega}\right)^2\left(\frac{Eh}{2\rho R_o}\right)\left(1 - 2\frac{k}{\omega}U^s\right)^{-1}$$

$$-\left(\frac{\rho_o h\omega^2}{2\rho}\right)\left(\frac{k}{\omega}\right)^2(R_o)\left(1 - 2\frac{k}{\omega}U^s\right)^{-1} \tag{3-45c}$$

$$\alpha_{22} = \frac{\sigma k}{2\omega R_o(1 - I)}\left(1 - \frac{2k}{\omega}U^s\right)^{-1} - \frac{k}{\omega R_o} \tag{3-45d}$$

Rearranging Equation (3-45) and defining $\chi = (k/\omega)\ (Eh/2\rho R_o)^{1/2}$, called phase velocity ratio, we have

$$\left[\left(1 - \frac{I}{1-\sigma^2}\right)\left(1 + \frac{i\sqrt{i}}{2} R_o\sqrt{\frac{\rho\omega}{\mu}}\right)\right]\chi^4 + \left\{\left[-\sigma + \sigma\frac{k}{\omega}U^s + \frac{1}{4}\right]\right.$$

$$- (1 - I)\left[\frac{\rho_o h\omega^2 R_o}{2\rho}\left(\frac{k}{\omega}\right)^2\left(1 + \frac{i\sqrt{i}}{2} R_o\sqrt{\frac{\omega\rho}{\mu}}\right)\right.$$

$$\left.\left. + \left(1 - 2\frac{k}{\omega}U^s\right)\frac{i\sqrt{i}}{2} R_o\sqrt{\frac{\omega\rho}{\mu}}\right]\right\}\chi^2$$

$$-\frac{1}{4}(1-\sigma^2)\left[\left(1 - 2\frac{k}{\omega}U^s\right) + \frac{\rho_o h\omega^2}{2}\left(\frac{k}{\omega}\right)^2 R_o\right] = 0 \qquad (3\text{-}46)$$

and

$$1 - I = 0 \qquad (3\text{-}46a)$$

Four new parameters are defined as follows:

$$\gamma = \frac{U^s}{\left(\frac{Eh}{2\rho R_o}\right)^{1/2}} \qquad (3\text{-}47a)$$
steady-state velocity parameter

$$\beta = \frac{(1-\sigma^2)\rho_o h}{2\rho R_o} \qquad (3\text{-}47b)$$
axial inertial parameter

$$\eta = \frac{\rho_o R_o^2 \omega^2}{E} \qquad (3\text{-}47c)$$
radial inertial parameter

$$z = R_o\left(\frac{\rho\omega}{\mu}\right)^{1/2} \qquad (3\text{-}47d)$$
viscosity parameter

Equation (3-46) becomes dimensionless:

$$\left[1 - \frac{z}{2\sqrt{2}}(1-i)\right](1-\eta)\chi^4 + \gamma\left[\frac{z}{\sqrt{2}} + \sigma - \frac{iz}{\sqrt{2}}\right]\chi^3 + \left\{\left[1 - \frac{z}{2\sqrt{2}}(1-i)\right]\right.$$

$$\left.\left[1+\beta\left(\eta - \frac{1}{1-\sigma^2}\right) - \frac{1}{4}\eta(1-\sigma^2) - \left(\frac{3}{4}+\sigma\right)\right]\right\}\chi^2 + \frac{(1-\sigma^2)}{2}\gamma\chi - \frac{1-\sigma^2}{4} = 0 \qquad (3\text{-}48)$$

This is a fourth-order equation containing complex numbers. One way to solve this equation is to separate its real part from the imaginary part. The result of this separation is

$$\left(K_1^4 - 6K_1^2K_2^2 + K_2^4\right)\left(1 - \frac{z}{2\sqrt{2}}\right)(1-\eta) - \left(4K_1^3K_2 - 4K_1K_2^3\right)\frac{z}{2\sqrt{2}}(1-\eta)$$

$$+ \left(K_1^3 - 3K_1K_2^2\right)\left(\frac{z}{\sqrt{2}} + \sigma\right)\gamma + \frac{\gamma z}{\sqrt{2}}\left(3K_1^2K_2 - K_2^3\right) + \left(K_1^2 - K_2^2\right)\left(1 - \frac{z}{2\sqrt{2}}\right)$$

$$\left[1+\beta\left(\eta - \frac{1}{1-\sigma^2}\right) - \frac{1}{4}\eta\left(1-\sigma^2\right) - \left(\frac{3}{4}+\sigma\right)\right]$$

$$- \left(2K_1K_2\right)\frac{z}{2\sqrt{2}}\left[1+\beta\left(\eta - \frac{1}{1-\sigma^2}\right) - \frac{1}{4}\eta\left(1-\sigma^2\right) - \left(\frac{3}{4}+\sigma\right)\right]$$

$$+ K_1\left(\frac{1-\sigma^2}{2}\right)\gamma - \frac{1-\sigma^2}{2} = 0 \qquad (3\text{-}49)$$

and

$$\left(K_1^4 - 6 K_1^2 K_2^2 + K_2^4\right)\left(\frac{z}{2\sqrt{2}}\right)(1 - \eta)$$
$$+ \left(4 K_1^3 K_2 - 4 K_1 K_2^3\right)\left(1 - \frac{z}{2\sqrt{2}}\right)(1 - \eta)$$
$$+ \left(3 K_1^2 K_2 - K_2^3\right)\left(\frac{z}{\sqrt{2}} + \sigma\right)\gamma - \frac{\gamma z}{\sqrt{2}}\left(K_1^3 - 3 K_1 K_2^2\right)$$
$$+ \left(K_1^2 - K_2^2\right)\frac{z}{2\sqrt{2}}\left[1 + \beta\left(\eta - \frac{1}{1-\sigma^2}\right) - \frac{1}{4}\eta\left(1-\sigma^2\right) - \left(\frac{3}{4} + \sigma\right)\right]$$
$$+ \left(2K_1K_2\right)\left(1 - \frac{z}{2\sqrt{2}}\right)\left[1 + \beta\left(\eta - \frac{1}{1-\sigma^2}\right) - \frac{1}{4}\eta\left(1-\sigma^2\right) - \left(\frac{3}{4} + \sigma\right)\right]$$
$$+ K_2\left(\frac{1 - \sigma^2}{2}\right)\gamma = 0 \qquad (3\text{-}50)$$

where

$$K_1 = \frac{k_1}{\omega}\left(\frac{Eh}{2\rho R_o}\right)^{1/2}$$ is the phase parameter

and

$$K_2 = \frac{k_2}{\omega}\left(\frac{Eh}{2\rho R_o}\right)^{1/2}$$ is the attenuation parameter

4. NUMERICAL SOLUTION

There are several ways to solve the two simultaneous fourth-order Equations (3-49) and (3-50). However, it seems that the combination of the root-mean-square error and contour plot technique is the most suitable method.

A pair of values, one for the phase velocity parameter, K_1, and another for the attenuation parameter, K_2, are arbitrarily chosen and substituted into the two equations. Since it is unlikely that the exact solution will be chosen, two errors, e_1 and e_2, result from these equations. If the root-mean-square of these two errors be called e, then $e = \left(e_1^2 + e_2^2\right)^{1/2}$. The better the estimate is, the smaller the root-mean-square error will be. The set of K_1 and K_2 values causing e = 0 is the solution of the equations for the particular assigned values of the four independent parameters.

Because of the limitations of the computer, the location of the point where e = 0 must be determined by trial and error. The way in which the computer program performs is, in the region of K_1 and K_2 specified for the trial, hundreds of root-mean-square errors are calculated; then, through these points, contour lines of different values of root-mean-square error are drawn. The point having zero root-mean-square value is determined by the programmer from the K-graph which is the output of the designated K_1 and K_2 computer program. Since it is difficult to make the right choice of the ranges of K_1 and K_2 on the first estimate, the program must be run twice or more for one set of K_1 and K_2 values. Because of the combined effort of the computer and human judgement, the calculation becomes quite tedious and time consuming.

Three K-graphs are attached: Figure 1 shows the coordinates of one solution of $K_1 = 1.92$ and $K_2 = 0.01$ for $\beta = 4.0$, $\eta = 0.1$, $\gamma = 10^{-6}$, and $z = 10$. In Figure 2, there are two solutions of points $K_1 = 0.01$, $K_2 = 0.93$; and $K_1 = 0.77$, $K_2 = -0.01$ for $\beta = \eta = \gamma = 0$ and $z = 0.1$. The other two points are the images of the first two. Since $-K_1$ and $-K_2$ will cause exactly the same numerical values of the e_1 and e_2 as $+K_1$ and $+K_2$, there is always an image point symmetrical about the origin where $K_1 = K_2 = 0$. Therefore, they are not to be considered as the solutions. Figure 3 shows no solution point in this region of K_1 and K_2 for the specified parameter values.

After the values of K_1 and K_2 are found, they are fed into another computing program called "FEDLIN" to calculate the velocity profiles and other needed values. The program has a subroutine for plotting graphs of different combinations of the independent variables or parameters. Since the imaginary parts of the final answer have no physical meaning, only the real parts of the complex values of the pressure and velocities are used in the graphs.

By inspecting Equations (3-21) and (3-22), then (3-39), there is a singularity at the center of the pipe for both v_x' and v_x'. The computing program for calculating and plotting the velocity profiles is, therefore, written to cover the radius ratio, r/R_o from δ to 1.0, where δ is an arbitrary small distance from the center line of the pipe. Since the flow is continuous and finite throughout the region, it is possible to assign a value of δ as small as we please. At the present, its value is 0.1. A set of graphical results that includes different kinds of velocity profiles and pipe wall displacements is included at the end of this report.

Table 1 shows the values of the constants and parameters used for making the calculations with the FEDLIN program.

Table 1. Computer Input Data for FEDLIN Program for Figures Indicated

Term	Unit	Value*	
		Group 1	Group 2
E	Psi	3.0×10^7	3.0×10^7
σ	-	0.3	0.3
$p_{x=0}$	Psi	40.0	40.0
P	Psi	2.0	2.0
x	Feet	20.0	15.96
ω	Rad/Sec	0.1885	7.0
t	Sec	1.0	0.224
K_1	-	0.241	0.16
K_2	-	0.308	0.135
z	-	99.2	10.0
η	-	1.37×10^{-11}	0.01
γ	-	0.011	1.0×10^{-4}
β	-	0.287	1.0

*Group 1: Figures 4a, 5a, 6, 7, 9, and 10
Group 2: Figures 4b, 5b, 8, 11a, and 11b

5. CONCLUDING REMARKS AND RECOMMENDATIONS

Because of the complex nature of the computing procedure, only a portion of the results have been generated to date. These computer outputs have shown that:

1. The axial velocity profile of the perturbed state is extremely sensitive to the viscosity of the flowing fluid. The boundary layer grows rapidly when the viscosity increases.

2. The magnitude of the perturbed axial velocity is very small in comparison with that of the steady-state velocity. This is consistent with the assumption of oscillation with small amplitude.

3. The steady flow does affect the phase velocity in the fluid. However, more data are needed to show the exact relationship.

4. Additional computer data is required to show the interrelationships between velocity profiles and various parameters and to compare with the results of wave propagation through fluid at rest discussed in Part II.

This study has covered the influence of pipe wall inertia in both the axial and radial directions. However, the results are limited to an incompressible fluid of such small viscosity that $(\mu U^s/Eh)(\lambda/R_o) \ll 1$ for a long wave length disturbance. Because of the severity of this assumption, it is recommended that the study be extended to the more practical case of ignoring the small viscosity effect but taking the compressibility of the fluid into full account. Instead of the Navier-Stokes equation, this proposed approach would use the well-known wave equation coupled with Donnel's axisymmetrical cylindrical shell equation.

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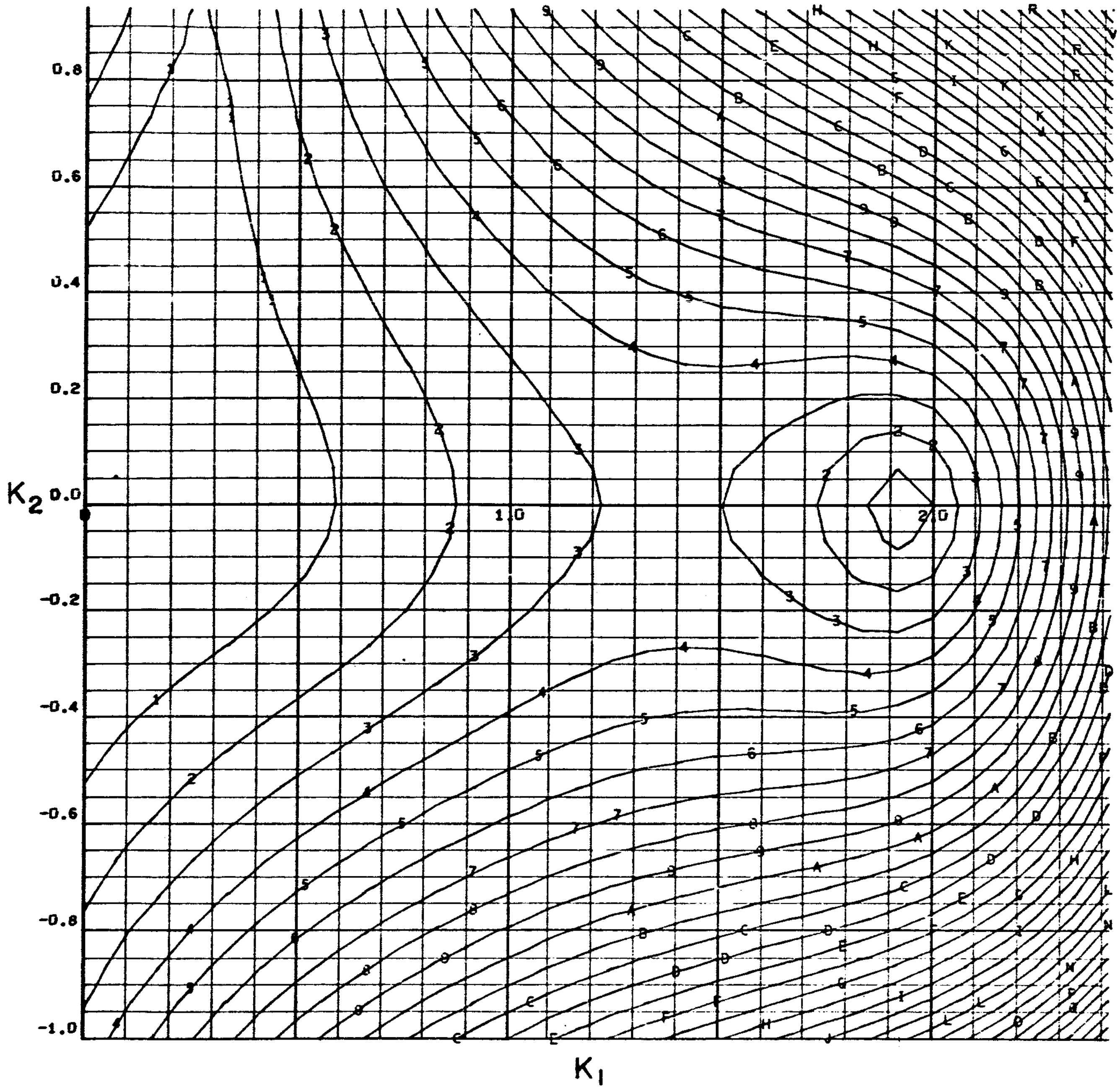

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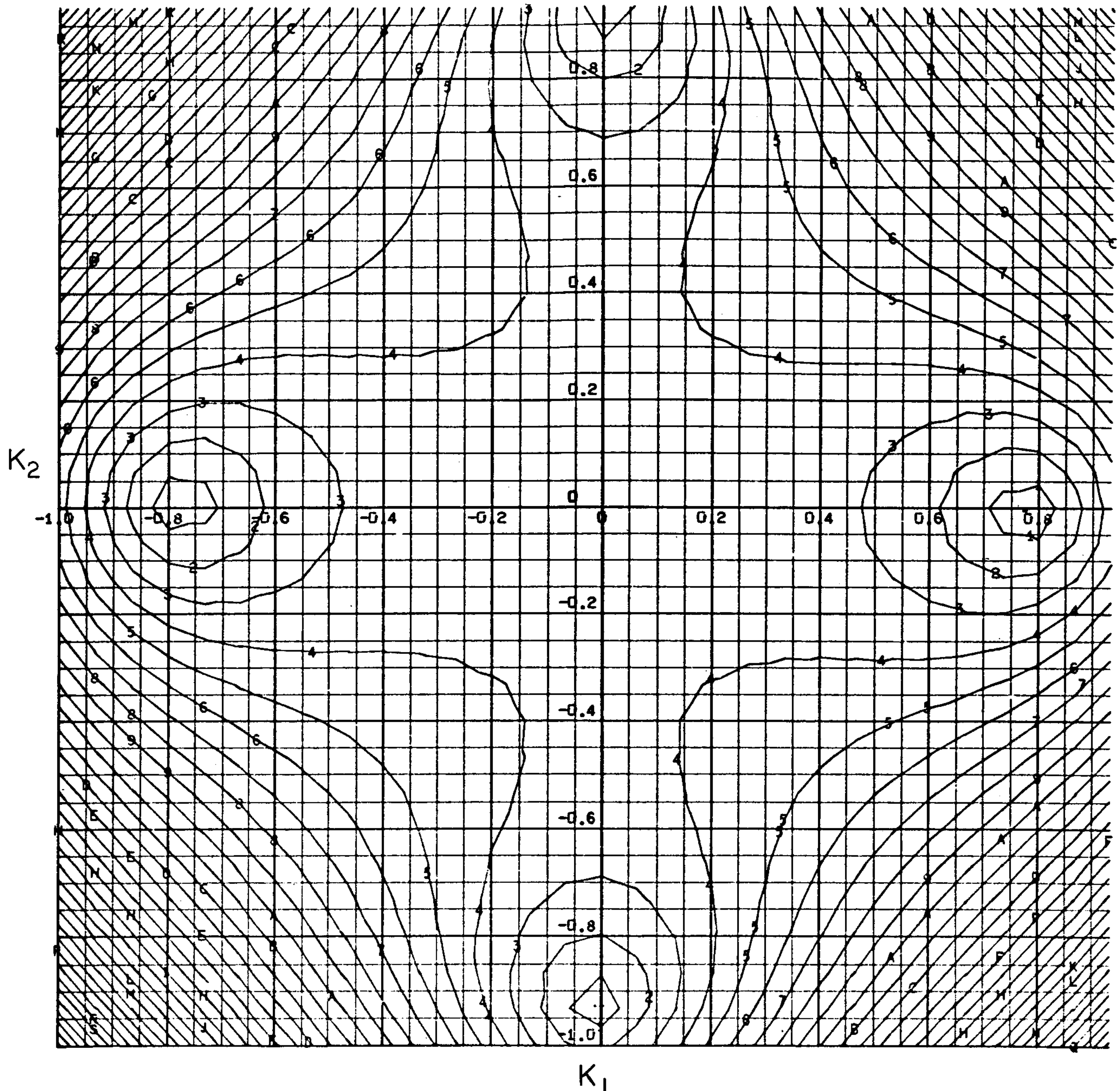
3	12.873170	C	51.492676	L	90.112179	U	128.731680
4	17.164226	D	55.783732	M	94.403234	V	133.022735
5	21.455282	E	60.074788	N	98.694290	W	137.313791
6	25.746339	F	64.365845	O	102.985346	X	141.604847
7	30.037395	G	68.656900	P	107.276402	Y	145.895903
8	34.328452	H	72.947956	Q	111.567457		
9	38.619508	I	77.239012	R	115.858513		

Figure 1. Error Contour-plot Solution for Equations (3-52) and (3-53), Showing One Solution



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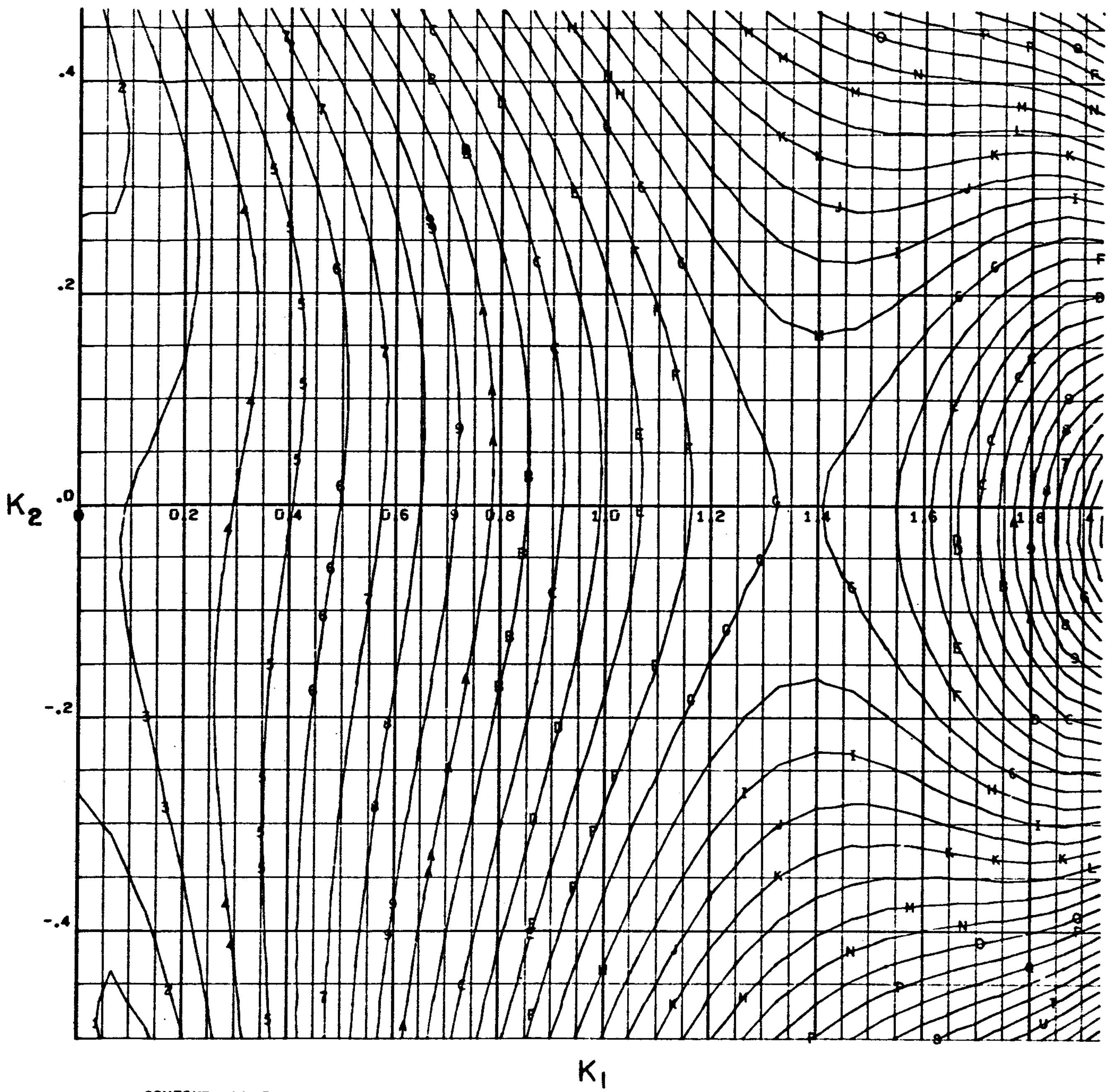
3	0.395456	C	1.581825	L	2.768194	U	3.954563
4	0.527275	D	1.713644	M	2.900013	V	4.086381
5	0.659094	E	1.845463	N	3.031831	W	4.218201
6	0.790913	F	1.977282	O	3.163651	X	4.350019
7	0.922731	G	2.109100	P	3.295469	Y	4.481838
8	1.054550	H	2.240919	Q	3.427288		
9	1.186369	I	2.372738	R	3.559107		

Figure 2. Error Contour-plot Solution for Equations (3-52) and (3-53), Showing Two Solutions and Their Images



CONTOUR IDENT.

1	0.173072	A	1.730719	J	3.288366	S	4.846013
2	0.346144	B	1.903791	K	3.461438	T	5.019084
3	0.519216	C	2.076862	L	3.634510	U	5.192157
4	0.692288	D	2.249935	M	3.807582	V	5.365229
5	0.865360	E	2.423007	N	3.980654	W	5.538301
6	1.038431	F	2.596079	O	4.153726	X	5.711372
7	1.211503	G	2.769150	P	4.326797	Y	5.884444
8	1.384575	H	2.942222	Q	4.499869		
9	1.557647	I	3.115294	R	4.672941		

Figure 3. Error Contour-plot Solution for Equations (3-52) and (3-53), Showing No Solution

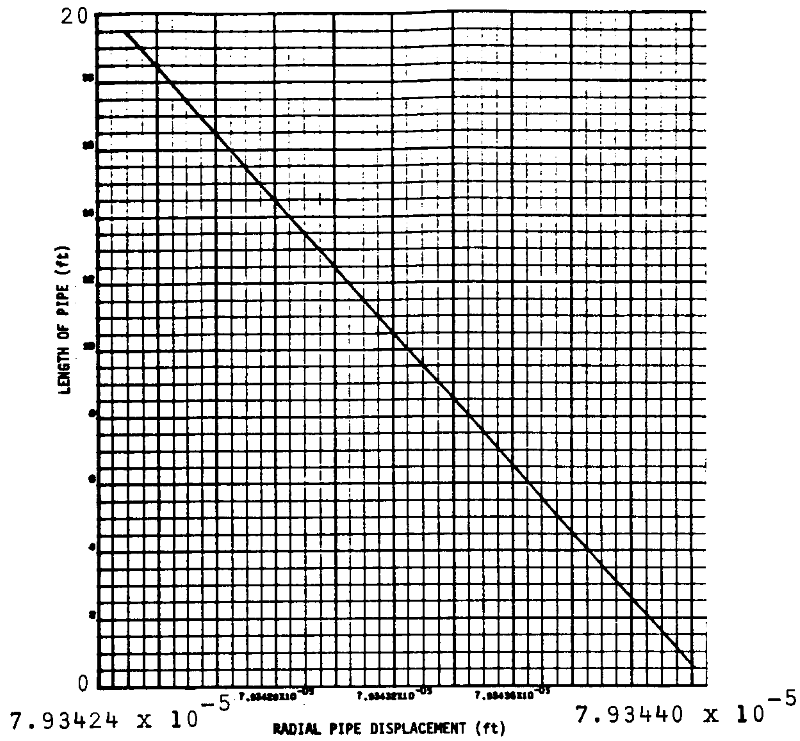


Figure 4a. Radial Displacement of the Pipe Wall

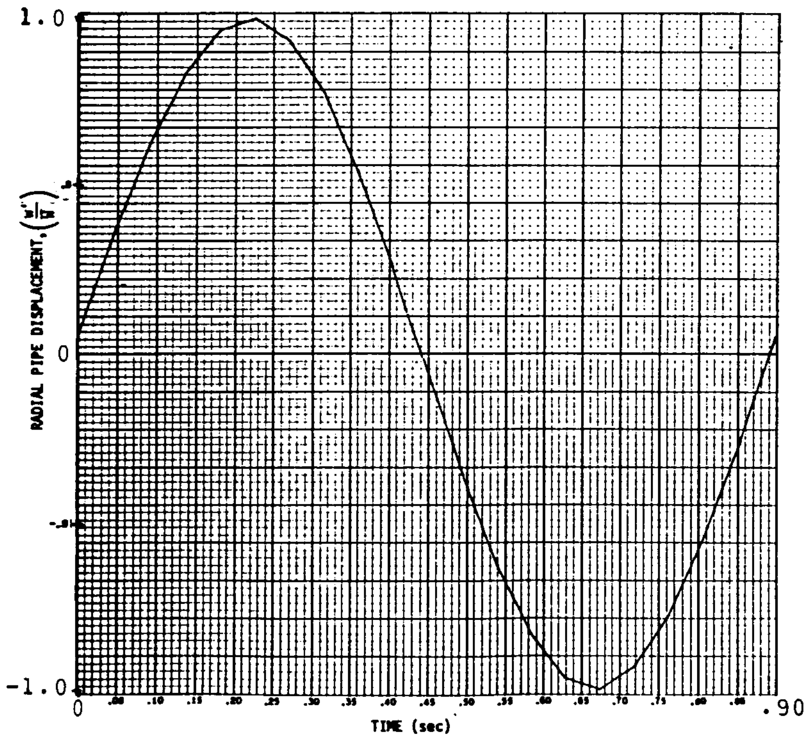


Figure 4b. Radial Oscillation of the Pipe Wall

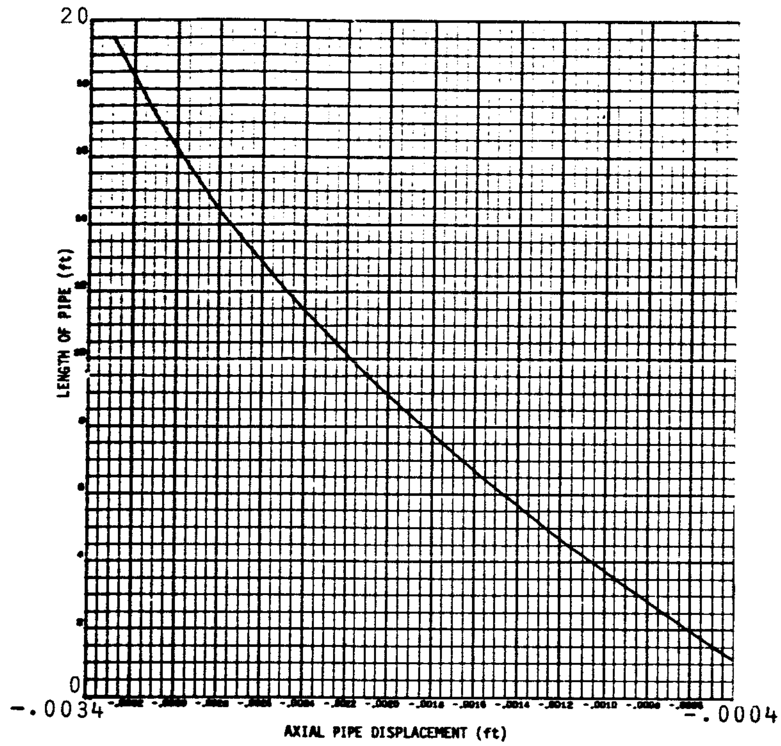


Figure 5a. Axial Displacement of the Pipe Wall

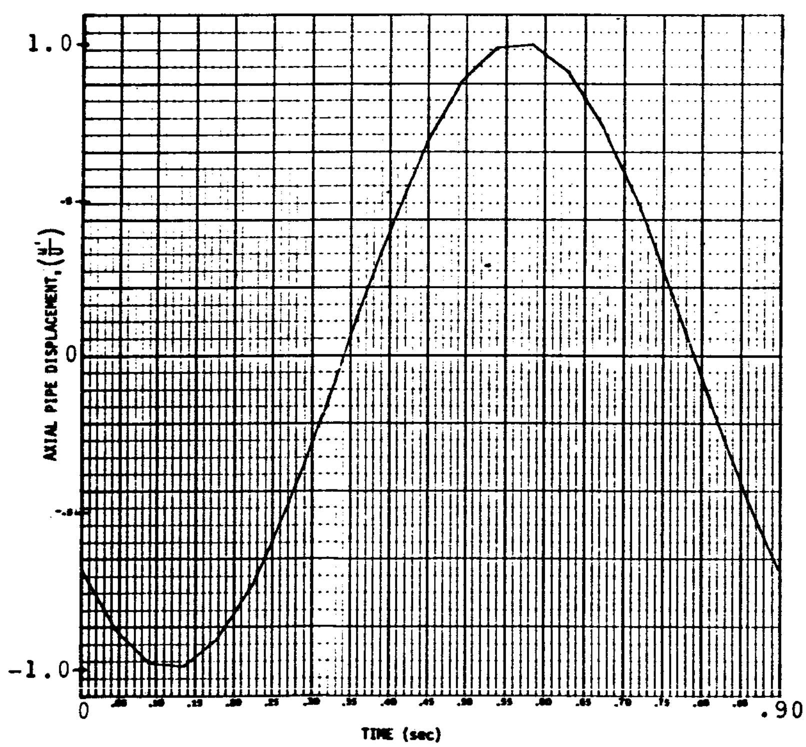


Figure 5b. Axial Oscillation of the Pipe Wall

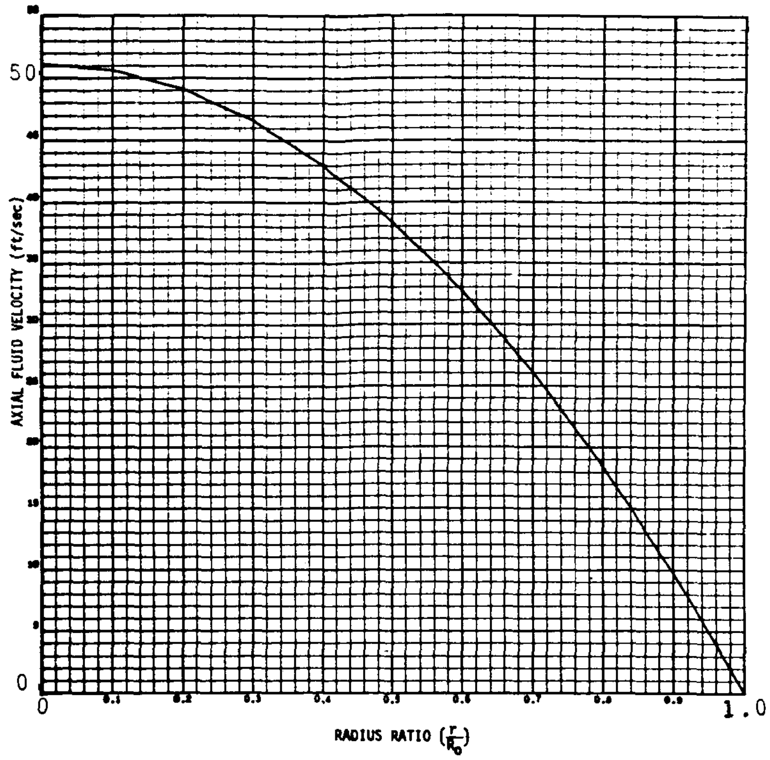


Figure 6. Axial Steady-State Velocity Profile

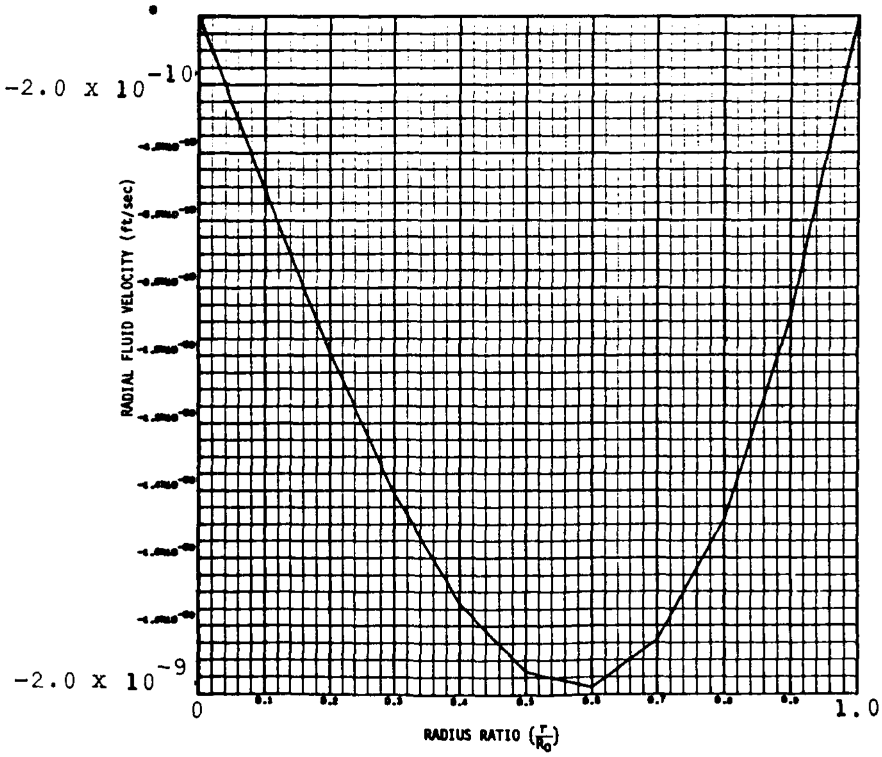


Figure 7. Radial Steady-State Velocity Profile

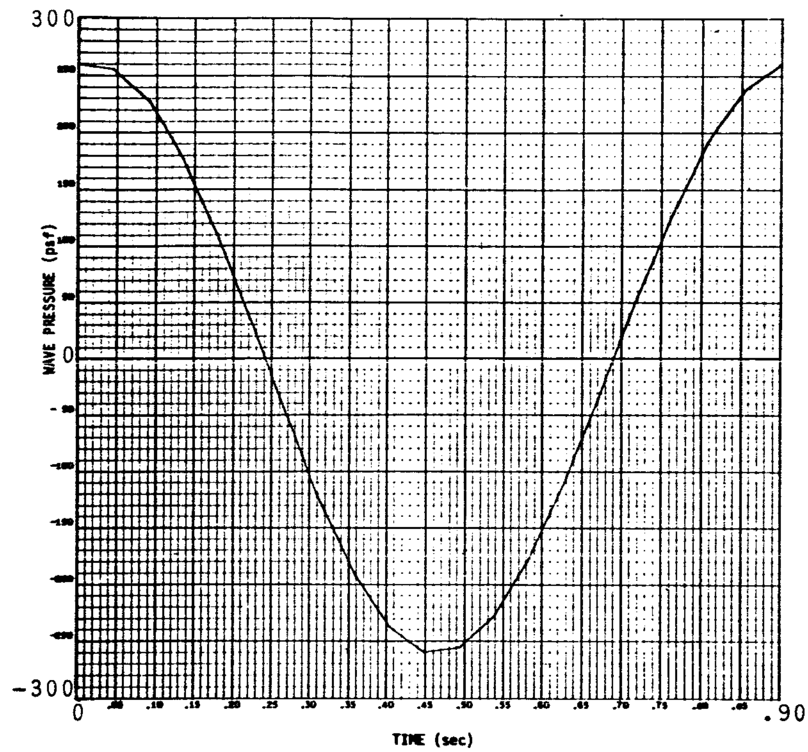


Figure 8. Pressure Oscillation Curve

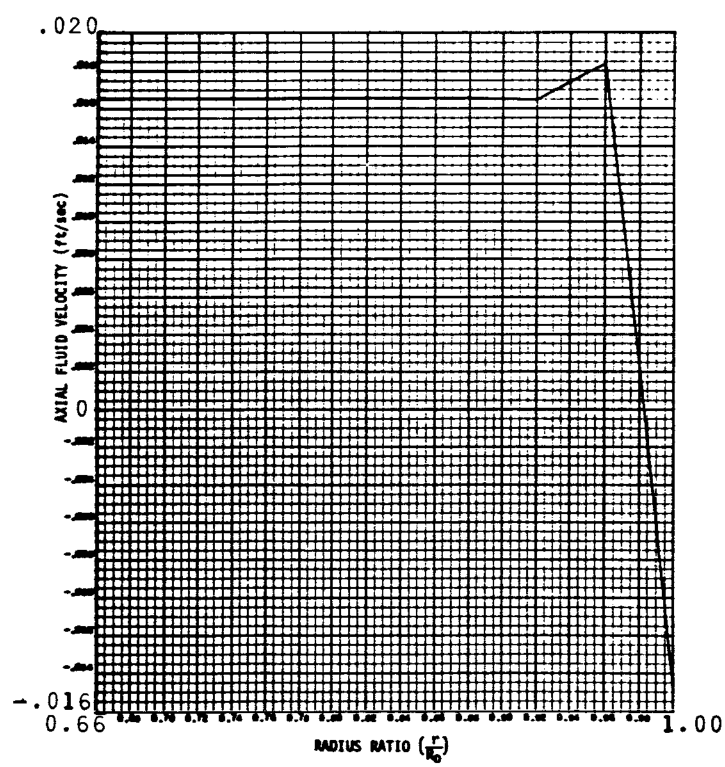


Figure 9. Axial Perturbed Velocity Profile Near the Pipe Wall

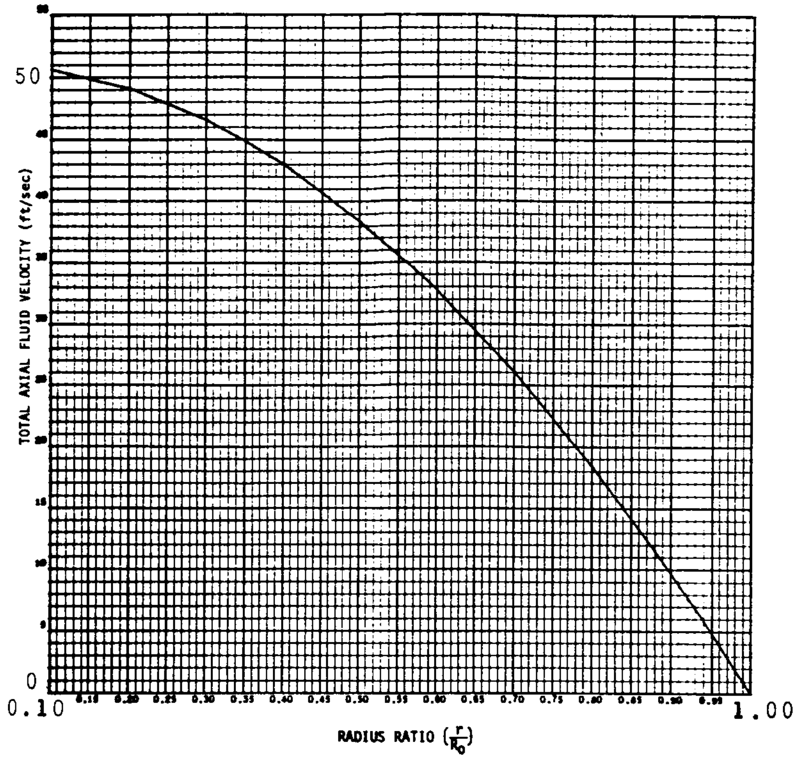


Figure 10. Axial Velocity Profile for the Superimposed Flow

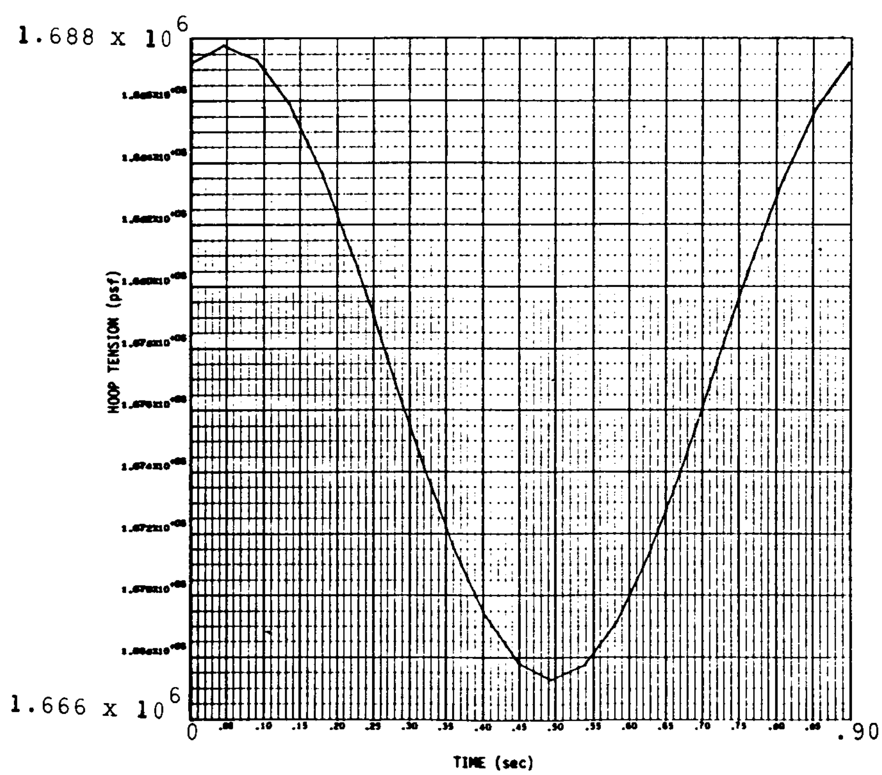


Figure 11a. Hoop Tension Curve

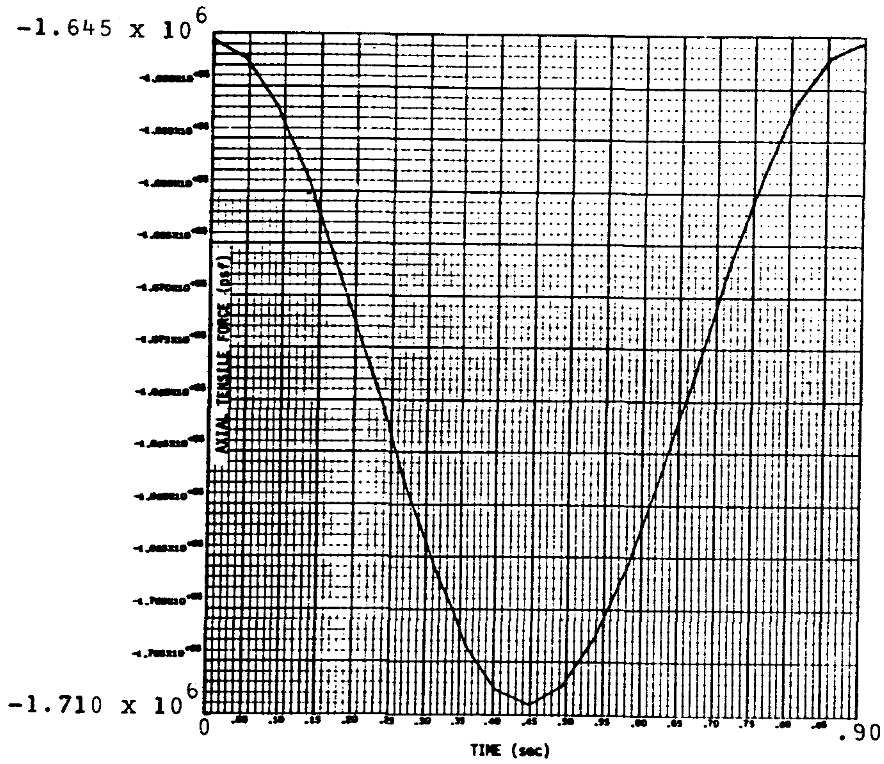


Figure 11b. Tensile Force in the Axial Direction